REGULARITY OF LABEL-SEQUENCES UNDER CONFIGURATION TRANSFORMATIONS

BY

T. N. E. GREVILLE(1)

1. **Introduction.** It is the purpose of this paper to develop a set of transformations to be applied to a label-sequence, or a set of such sequences, yielding a single, new sequence, and to show that certain essential properties of the sequences are almost always preserved under such transformations. This result, like other less general theorems previously given by other writers, provides a new kind of justification for the use of the classical rules for the combination of probabilities. These transformations are defined by setting up certain correspondences between configurations of elements in the original sequences and single elements in the sequence resulting from the transformation. The set of transformations considered will be shown to include as special cases the four fundamental operations in terms of which the rules for the combination of probabilities have been formulated analytically by von Mises(2), as well as the operations used by Copeland, Reichenbach, and Popper in defining classes of canonical sequences.

The concept of the configuration transformation was first suggested by a consideration of the problem of "sets of games." A limited application of the principle to certain simple cases was made in an earlier paper in which it was stated, on the authority of von Mises himself, that his methods were not applicable to the formulation of such problems(3). He has since pointed out(4), however, that the general "set-of-games" problem can be handled by means of a suitable combination of the operations of selection, mixing, and combination. A more general class of problems can be dealt with through the transformations of this paper; and, in any case, it is convenient to work with a single type of operation which includes all the others as special cases. Moreover, it simplifies the treatment of complicated problems to express them in

Presented to the Society, September 8, 1942, and also April 7, 1939 under the title *Invariance of the admissibility of variates under certain general types of transformations*; received by the editors January 29, 1943, and, in revised form, March 31, 1943.

(1) A dissertation submitted in partial fulfillment of the requirements for the degree of doctor of philosophy in the University of Michigan. The writer wishes to express his appreciation to Professor A. H. Copeland for his valuable assistance in the preparation of this paper, and to Professor R. von Mises for permitting him to read the manuscript of an unpublished paper [3]. Numbers in brackets refer to literature cited at end of paper.

(2) von Mises [1, pp. 75–94].

(3) Greville [1, p. 410]; von Mises [1, pp. 108–109].

(4) von Mises [2, pp. 195–197; 3].

terms of a single operator rather than a large (and frequently infinite) number.

My results as to the invariance of certain properties of sequences under configuration transformations include as special cases most of the theorems previously proved on the existence of the collective. For the methods used in arriving at these results, I am much indebted to Kolmogoroff, Feller, and Doob(5).

2. **The configuration transformation.** Let $[\pi_1, \pi_2, \cdots]$ be a set (finite or denumerable) of arbitrary sets, each containing more than one element, which will be called *label-spaces*. Let $x=[x_1, x_2, \cdots]$ be a set of infinite label-sequences, $x_i=x_i^{(1)}, x_i^{(2)}, \cdots$, each sequence x_i being composed of elements of the set π_i.

A transformation on the set x will be defined by means of a *configuration function*(6) $f(c)$, in which c denotes a finite configuration of elements of the product-space $\pi=\pi_1\times\pi_2\times\cdots$. It is assumed that $f(c)$ is defined for some, but not necessarily all, finite configurations of elements of π. The "values" of $f(c)$ may be any arbitrary elements.

From the sequence of configurations: $c_1, c_2, \cdots$, in which c_r is the configuration formed by the first r elements of all the sequences x_i, select the subsequence $c_{r_1}, c_{r_2}, \cdots$ consisting of those configurations c_r for which $f(c_r)$ is defined. Finally, form the sequence $y=T(x)=f(c_{r_1}), f(c_{r_2}), \cdots$. This defines the configuration transformation T. The label-space ρ, consisting of all possible "values" of $f(c)$, will be assumed to contain more than one element.

As an example of a configuration transformation, consider two players simultaneously throwing dice, who agree that A is to win a bet if he throws a one before B throws a six. Otherwise, B wins; but if both throw their numbers on the same toss, it is not counted. Here, π_1 and π_2 each consist of the integers 1 to 6, and π consists of all the possible pairs of such integers, while ρ consists of the two labels A and B. The configurations for which the function $f(c)$ is defined are those finite sequences of pairs of integers 1 to 6 terminating with a pair whose first member is 1 *or* whose second member is 6, but not both. $f(c)$ has the value A in the former case, B in the latter.

The four fundamental operations of von Mises, place selection, partition, mixing, and combination, are all particular types of configuration transformations. The first three are all operations on a single sequence. A place selection can be defined(7) by means of a configuration function $g(c)$ defined for all finite configurations, and restricted to the values 1 and 0. A sequence x' is selected from the sequence x by adopting the rule that any element $x^{(i)}$ is

(5) Kolmogoroff [1], Feller [1], Doob [1].

(6) Configuration functions have previously been used by Wald [1], Doob [1], Feller [1], and Ville [1], as a means of defining position selections.

(7) Wald [1, pp. 38–39]; Doob [1, pp. 364–365]; Feller [1, p. 89]; Ville [1, pp. 41–42]. See also von Mises [1, p. 75].

selected if and only if $g(x^{(1)}, x^{(2)}, \cdots, x^{(i-1)}) = 1$. This operation can be regarded as a configuration transformation by merely including the element selected in the configuration, considering the label-spaces π and ρ as identical.

The operation of mixing(8) consists in grouping together certain sets of labels and agreeing to represent all the members of a given group or set by a single label. The operation of partition consists(9) in selecting from a sequence x a subsequence composed of all those elements belonging to some subset γ of π. The operation of combination(10) consists in combining a set (finite or denumerable) of sequences $[x_1, x_2, \cdots]$ into a single sequence whose elements are points in a multi-dimensional space having as coordinates the corresponding elements of the individual sequences. All these operations are clearly particular cases of the configuration transformation.

3. **A measure theory.** Every set of label-sequences $x = [x_1, x_2, \cdots]$ may be regarded as a point of the infinite product-space $\Pi = \pi \times \pi \times \cdots$. In order to facilitate the study of the behavior of sequences under configuration transformations, a measure theory on Π will be developed. It will be assumed that there exists a field Φ, consisting of subsets of π and including π itself, and a non-negative, additive set function $p(\gamma)$, defined on Φ, such that $p(\pi) = 1$. It will be assumed further that $p(\gamma)$ is absolutely continuous: that is, if there is a monotone sequence $\gamma_1 \supset \gamma_2 \supset \cdots$ approaching the null set as limit, then

$$\lim_{k \to \infty} p(\gamma_k) = 0.$$

According to a well known extension theorem of Banach(11), a set function $p(\gamma)$ having the above properties can be extended into an absolutely additive set function defined on a Borel field Φ'(12). The function so obtained will be called the Borel extension(12) of the function $p(\gamma)$.

We shall follow Kolmogoroff in calling the set $g = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n \times \pi \times \pi \times \cdots$, where each λ_i is an element of π, a *cylinder-set*(13) on the space Π; and we define $P(g) = p(\lambda_1)p(\lambda_2) \cdots p(\lambda_n)$. Lomnicki and Ulam(14) have shown that a set function thus defined can be extended into an absolutely additive (and therefore absolutely continuous) measure function defined on a Borel field F consisting of subsets of Π. Obviously, $P(\Pi) = 1$.

If the sequences $x_1, x_2, \cdots$ are independent, it is possible to begin by

(8) von Mises [1, p. 76].

(9) von Mises [1, p. 86].

(10) von Mises [1, p. 94].

(11) See Kolmogoroff [1, pp. 15–16], Feller [1, p. 90].

(12) A Borel field is one having the property that every set which is the sum of a denumerable infinity of sets of the field belongs to the field. See Kolmogoroff [1, pp. 15–16].

(13) Kolmogoroff [1, p. 25] uses the term *cylinder-set* in a somewhat broader sense. We shall therefore designate as a *simple cylinder-set* a set of the simple product type used here. See also Feller [1, p. 90].

(14) Lomnicki and Ulam [1, p. 252]. Kolmogoroff [1, p. 27] has also proved this theorem for the case of a Euclidean space. See also Doob [1].

assuming the existence on each space π_i of a set function $p_i(\gamma_i)$, defined on a field Φ_i, having the required properties. A unique measure function $p(\gamma)$ on a field Φ is then obtained by the same process used in developing the function $P(g)$. $p(\gamma)$ automatically possesses the necessary properties.

In order for this measure on Π to be of use in the study of configuration transformations, it is necessary to limit consideration to configuration functions $f(c)$ which are measurable. Consider a finite product-set $s=\lambda_1\times\lambda_2\times\cdots\times\lambda_n$(15) on the n-dimensional product-space $\rho\times\rho\times\cdots\times\rho$. Let E_s denote the point set on Π consisting of sets of sequences $x=[x_1, x_2, \cdots]$ such that the configuration formed by the first n elements of the corresponding sequence y belongs to s. Further, let E_{sm} denote the subset of E_s consisting of those points x for which the configuration c associated with the element $y^{(n)}$ (through the relation $f(c)=y^{(n)}$) contains m elements. If there exists a field Ψ, consisting of subsets of ρ including ρ itself, and such that, for every such set s formed from elements of Ψ and for every m, the corresponding set E_{sm} belongs to the field F, the configuration function $f(c)$ is said to be *measurable* on Ψ, and the transformation T is called a *measurable transformation*. In the case of a selection, this definition can be shown to be equivalent to that given by Feller(16).

4. **The regularity theorem.** In studying the behavior of sequences under configuration transformations, we are interested in the frequency with which certain sets of elements appear in the transformed sequence. We shall develop a measure function on the transformed label-space ρ, based on the structure of the transformation T, which will indicate what frequencies are to be expected for these sets on an a priori basis. The measure theory already developed on Π will make it possible to determine whether our expectations are satisfactorily met. In general we shall consider this to be the case if the set of untransformed sequence-sets not producing the proper frequencies in the transformed sequence is of measure zero on Π. In order to secure a suitable measure function on the transformed space, it is necessary to impose certain further (although as shown later, not too severe) limitations on the transformations which may be considered.

A measurable configuration transformation T will be called *endometric* if there exists an additive set function $q(\delta)$, defined for every set δ of Ψ, such that $q(\rho)\leqq 1$, and having the property that, for every finite product-set $s=\lambda_1\times\lambda_2\times\cdots\times\lambda_n$ formed from sets of Ψ,

$$P(E_s) \leqq q(\lambda_1)q(\lambda_2)\cdots q(\lambda_n). \tag{1}$$

The function $q(\delta)$ will be called the *expected frequency* of the set δ. Let

(15) We shall always use the ordinary multiplication sign to indicate a multi-dimensional product-set and the dot to indicate the set-product or common part of a group of sets.

(16) Feller [1, p. 90].

$\theta_n(\delta, y)$ denote the number of elements, out of the first n elements of y, which belong to δ.

THEOREM 1. *If T is an endometric configuration transformation, and δ is a set of the field Ψ, then, for almost every point x of* Π, *either the sequence $y = T(x)$ terminates after a finite number of elements, or else*(17)

$$\lim_{N\to\infty} N^{-1}\theta_N(\delta, y) = q(\delta). \tag{2}$$

Let $M_{n,k}$ denote the set of points of Π for which the sequence $y = T(x)$ contains at least n elements and exactly k of the first n elements belong to δ. Let η denote the set $\rho - \delta$. Then, since T is endometric, $q(\rho) = q(\delta) + q(\eta) \leqq 1$. Hence $q(\eta) \leqq 1 - q(\delta)$. It follows from (1) that

$$P(M_{n,k}) \leqq C_{n,k}[q(\delta)]^k[1 - q(\delta)]^{n-k}.$$

By a well known property of the binomial expansion(18), the measure of the set

$$\sum_{n=N}^{\infty} \sum_{|k/n - q(\delta)| > \epsilon} M_{n,k}$$

approaches zero as N tends to infinity, for every positive ϵ. Therefore the point set on Π for which the upper or the lower limit of $N^{-1}\theta_n(\delta, y)$ differs from $q(\delta)$ by more than ϵ has the measure zero.

This theorem leaves something to be desired, since it implies that a transformed sequence terminating after a finite number of elements is quite as satisfactory as an infinite sequence having the expected frequency for the set δ. By imposing certain additional requirements on T, this defect can be eliminated. A measurable transformation T will be called *complete* if, for almost every point x of Π, the sequence $y = T(x)$ contains an infinite number of elements. A transformation will be called *holometric* if it is both endometric and complete. A point x on Π such that the sequence $y = T(x)$ does not terminate, and such that (2) holds for *every* set δ of Ψ will be called *a regular point under T*. A set function $f(a)$ defined on a field A will be called *denumerably ordered* if there exists a denumerable set of sets b_k $(k = 1, 2, \cdots)$, all contained in A, such that for every set a of A(19),

$$f(a) = \underset{b_k \supset a}{\text{G.L.B.}}\, f(b_k) = \underset{b_k \subset a}{\text{L.U.B.}}\, f(b_k). \tag{3}$$

The following theorem follows immediately from Theorem 1 and from the above definitions:

---

(17) This theorem was proved by Feller [1, pp. 89–93] for the case in which T is a selection.

(18) Borel [1, p. 4] Feller [1, p. 93].

(19) This condition was used by Feller [1, p. 89]. G. L. B. denotes "greatest lower bound"; L. U. B., "least upper bound."

THEOREM 2. *If a configuration transformation T is holometric with respect to a function $q(\delta)$ which is denumerably ordered on a field Ψ, then almost every point of Π is regular under T.*

5. **Iteration of configuration transformations.** It follows from the definition of a configuration transformation that if $y=T(x)$ and $z=T'(y)$, T' being one-dimensional, then there exists a configuration transformation T'' such that $z=T''(x)$. Therefore, any finite sequence of configuration transformations in which all but the first are one-dimensional can be replaced by a single configuration transformation.

In order to discuss the behavior of y under T', it is necessary to define a satisfactory measure function on the label-space ρ. The requirement that T be endometric is not quite adequate, as it does not impose sufficient limitation on $q(\delta)$ to permit its use as a measure function. For this reason, another special type of configuration transformation will now be defined. An endometric transformation will be called *isometric* if, for every product-set $s=\lambda_1\times\lambda_2\times \cdots \times\lambda_n$ formed from sets of Ψ,

$$P(E_s) = q(\lambda_1)q(\lambda_2) \cdots q(\lambda_n). \tag{4}$$

It will be shown later that every holometric transformation is isometric (see Theorem 4). Therefore, in the case of a holometric transformation, the function $q(\delta)$ meets all the requirements for a satisfactory measure function, and can therefore be extended, as indicated in §4, into an absolutely additive set function defined on a Borel field Ψ. It is then possible to define a measure function $Q(d)$ on a field G consisting of subsets of the infinite product space $R=\rho\times\rho\times \cdots$.

Returning to the discussion of iterated transformations, it will now be shown that if T is holometric and T' is measurable, then T'' is measurable. Let σ denote the label-space associated with the sequence z, and let $S=\mu_1\times\mu_2\times \cdots \times\mu_n$ denote a finite product-set on the n-dimensional product-space $\sigma\times\sigma\times \cdots \times\sigma$. Further, let H_S denote the set of points y on R such that the configuration formed by the first n elements of z belongs to S; and let H_{S^m} denote the subset of H_S consisting of those points y for which the configuration c associated with the element $z^{(n)}$ contains m elements. Since T' is measurable, there exists a field Ξ composed of subsets of σ, such that for every m and for every product-set S formed from elements of Ξ, H_{S^m} is measurable on R. Now, there corresponds to every sequence y the set of sequence-sets x such that $T(x)=y$. Similarly, there corresponds to every set of points y on R a set of points x on Π. If K denotes any point set on R and L the corresponding point set on Π, we shall presently show that L is measurable provided K is measurable, and moreover $P(L)=Q(K)$. But if the set H_{S^m} be taken as the set K, the set L is the set E''_{S^m} associated with the configuration S under the transformation T''. This not only shows that T'' is measurable if T' is measurable, but also that if T' is endometric, complete, holometric, or isometric, then T'' has the same property.

To prove the above statement regarding corresponding point sets on the spaces R and Π, first consider the case in which K is a simple cylinder-set $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n \times \rho \times \rho \times \cdots$, each λ_i being an element of Ψ'. In this case, it follows from (4) that $P(L) = Q(K)$, since T, being holometric, is therefore isometric (see Theorem 4). Now, the measure function $Q(d)$ on the entire field G was secured by extension from its definition for simple cylinder-sets. Moreover, Lomnicki and Ulam have shown that this extension is unique([20]). As the two measure functions $P(g)$ and $Q(d)$ have the same laws of combination, the same equality must hold for all measurable sets.

A sequence of configuration transformations applied in succession to a sequence-set x, so that we have $T_n\{\cdots T_2[T_1(x)]\} = z$, will be called a *chain*. It follows, of course, that all the transformations of the chain, with the possible exception of T_1, are one-dimensional. The following theorem about chains is an immediate consequence of the above remarks.

THEOREM 3. *A chain of configuration transformations in which the final transformation is measurable, and all the others holometric, is equivalent to a single, measurable configuration transformation. If the final transformation of the chain is endometric, complete, holometric, or isometric, then the equivalent single transformation has the same property.*

6. **Classification of configuration transformations.** Consider the division of any configuration c for which $f(c)$ is defined into two segments or subconfigurations which shall have the following properties. The first segment shall either be null, or else shall constitute a configuration for which $f(c)$ is defined. The second segment shall not be vacuous but otherwise shall contain as few elements as possible. It is clear that such a division is always possible, and is unique. The two segments so obtained will be called the *initial segment* and the *final segment* of c. If a configuration function $f(c)$ is a function of the *final segment only* of c, the corresponding transformation T is called *symmetric.* The transformation associated with the "set-of-games" problem is evidently symmetric, as are all mixings, partitions, and combinations.

On the other hand, place selections are not, in general, symmetric. An example is a type of selection employed by various writers([21]) in defining special classes of canonical sequences, an operation which selects every element in a sequence which is preceded by a specified configuration. Here two configurations on which selections are based may overlap, in which case the transformation is not symmetric. It is easily verified that the class of symmetric transformations possesses the iterative property previously mentioned.

The following are the principal theorems regarding the classification of configuration transformations:

([20]) Lomnicki and Ulam [1, pp. 250–251].

([21]) Reichenbach [1, p. 291], Popper [1, p. 106], Ville [1, p. 70].

THEOREM 4. *Every holometric configuration transformation is isometric.*

Consider the finite product-set $s=\lambda_1\times\lambda_2\times\cdots\times\lambda_n$ in which each set λ_i is an element of the field Ψ associated with the transformation T; and let $\mu_i=\rho-\lambda_i$. Then if $s'=(\lambda_1+\mu_1)\times(\lambda_2+\mu_2)\times\cdots\times(\lambda_n+\mu_n)$, the corresponding point set $E_{s'}$ on Π has unit measure, since the only points x not contained in this set are those for which $T(x)$ terminates with fewer than n elements. Since T is complete, the set of these points is of measure zero. Now, the product s' can be expanded algebraically into a sum of 2^n mutually exclusive product sets, $s_1, s_2, \cdots, s_{2^n}$. In order to have the sum of their measures unity, the measure of each of the corresponding sets, $E_{s_1}, E_{s_2}, \cdots, E_{s_{2^n}}$ (of which one is the set E_s) must have its maximum value according to (1).

THEOREM 5. *Every measurable symmetric transformation is isometric.*

Condition (4) can be satisfied, for the case $n=1$, by defining $q(\delta)=P(E_\delta)$. Assume that (4) holds for all $(n-1)$-dimensional product-sets, and consider the set $s=\lambda_1\times\lambda_2\times\cdots\times\lambda_n$. Let s' denote the set $\lambda_1\times\lambda_2\times\cdots\times\lambda_{n-1}$. Now, the set $E_{s'm}$ can, by definition, be expressed in the form $K_m\times\pi\times\pi\times\cdots$, where K_m is a subset of the m-dimensional product-space $\pi\times\pi\times\cdots\times\pi$. Let E_δ denote the set of points x such that the first element of y is an element of δ, δ being any subset of ρ. Then, $E_{s'm}\cdot E_s=K_m\times E_{\lambda_n}$, where the latter expression represents the set of all points x (regarded as sequences of points of the space π) which can be formed by prefixing to any sequence x belonging to the set E_{λ_n} the configuration formed by the m elements constituting any point of K_m. It follows from the manner in which the function $P(g)$ was constructed that $P(K_m\times E_{\lambda_n})=P(E_{s'm})P(E_{\lambda_n})$. But $E_s=\sum_{m=1}^{\infty}E_{s'm}\cdot E_s$; and since, by definition, the sets $E_{s'm}$ for different values of m are mutually exclusive, $P(E_s)=\sum_{m=1}^{\infty}P(E_{s'm})P(E_{\lambda_n})=P(E_{s'})q(\lambda_n)$. This completes the induction.

THEOREM 6. *Every measurable selection is endometric*(22).

Since, in the case of a selection, the label-spaces π and ρ are identical, take $q(\gamma)=p(\gamma)$. First, consider the case $n=1$, where $s=\lambda_1$. If x belongs to E_{sm}, it follows from the definition of a selection that the first $m-1$ elements of x determine whether or not the mth element will be selected, and that the only role of the mth element, so far as membership in the set E_s is concerned, is in determining whether the element selected belongs to the set λ_1. Therefore, the set $E_{\pi m}$ can be expressed in the form $K_{m-1}\times\pi\times\pi\times\cdots$, where K_{m-1} is a subset of the $(m-1)$-dimensional space $\pi\times\pi\times\cdots\times\pi$. Then, $E_{sm}=K_{m-1}\times L$, where L denotes the infinite product-set $\lambda_1\times\pi\times\pi\times\cdots$: that is, the set of all sequences x having as the first element a point of λ_1.

(22) Feller [1, pp. 92–93] gives a similar proof for the case in which each component λ_i of s is either a particular set γ or the complement $\pi-\gamma$.

Then, $P(E_{sm})=P(E_{\pi m})p(\lambda_1)$. Therefore,

$$P(E_s) = \sum_{m=1}^{\infty} P(E_{sm}) = \sum_{m=1}^{\infty} P(E_{\pi m})p(\lambda_1) = P(E_\pi)p(\lambda_1) \leqq p(\lambda_1).$$

This proves the theorem for the case $n=1$.

Now suppose the theorem to be true for all $(n-1)$-dimensional product-sets and consider a product-set s of n dimensions. Let s' denote the product-set obtained by replacing the final element λ_n of s by π, and let s'' denote the $(n-1)$-dimensional product-set obtained by deleting this element. Then, by reasoning similar to that employed in the case $n=1$, $P(E_{sm})=P(E_{s'm})p(\lambda_n)$. Therefore,

$$P(E_s) = \sum_{m=1}^{\infty} P(E_{sm}) = \sum_{m=1}^{\infty} P(E_{s'm})p(\lambda_n) = P(E_{s'})p(\lambda_n).$$

Since $E_{s'}$ is clearly a subset of $E_{s''}$, $P(E_{s'})\leqq P(E_{s''})$, whence $P(E_s) \leqq P(E_{s''})q(\lambda_n)$. But, by hypothesis, (1) holds for the product-set s''. This completes the induction.

The properties of the four fundamental operations of von Mises are of particular interest. First considering measurability, every combination is measurable if there exists a measure function on π satisfying the requirements previously laid down, for the field Ψ can then be chosen as identical with Φ. The partition is measurable, provided the set γ on which the selection is based (see §2) belongs to the field Φ, for in this case the set γ together with all its subsets which are also elements of Φ constitutes a field Ψ which is identical with a subfield of Φ. The measure function $q(\delta)$ is defined by

$$q(\delta) = p(\delta)/p(\gamma).$$

The mixing is measurable provided there exists on the space ρ a field Ψ into which some subfield of Φ is transformed by the mixing. No general statement can be made regarding the measurability of place selections.

The mixing, partition, and combination, being all symmetric transformations, are, by Theorem 5, isometric when measurable. Every measurable mixing or combination is evidently complete. Finally, all measurable partitions are complete (unless $p(\gamma)=0$), since it is easily shown that the set of points x on Π containing only a finite number of elements of γ is of measure zero.

7. **Illustration.** An illustration of the application of configuration transformations is to be found in the definition given by Copeland[23] for an *admissible variate.* Let the space π consist of all real numbers, and let ρ consist of the two labels 0 and 1. Let $r_1, r_2, \cdots, r_k, n$ be a set of integers such that $0<r_1<r_2<\cdots<r_k\leqq n$, and let $I_1, I_2, \cdots, I_k$ be a set of real intervals,

[23] Copeland [1, pp. 543–547].

open on the left and closed on the right. The configuration function $f(c)$ will be defined for those and only those configurations in which the number of elements is an integral multiple of n. Its value is determined by the last n elements of the configuration, and is 1 if and only if the r_ith of these last n elements falls in the interval I_i for each i ($i=1, 2, \cdots, k$). We shall call any transformation T thus defined a *Copeland transformation*. If $F(t)$ is any monotonic function such that $F(-\infty)=0$ and $F(\infty)=1$, a label-sequence x is said to be an admissible variate with respect to the function $F(t)$ provided the relative frequency of the label 1 in the corresponding sequence $y=T(x)$ tends to the limit $\prod_{i=1}^{k}\int_{I_i}dF(t+0)$, for every possible choice of the integers $r_1, r_2, \cdots, r_k, n$ and of the intervals I_i.

Given any function $F(t)$ satisfying the above conditions, Copeland shows that there exists an admissible variate with respect to $F(t)$. We shall show that almost every variate x is admissible with respect to any such function $F(t)$. Let the field Φ consist of all real intervals I open on the left and closed on the right, and all finite sums of such intervals; and for each such interval I, let $p(I)=\int_I dF(t+0)$. Consider the Copeland transformation T resulting from a particular choice of the integers $r_1, r_2, \cdots, r_k, n$ and the intervals I_i. This transformation is clearly measurable, and being symmetric and complete, is therefore holometric. By Theorem 2, almost every sequence x is regular under any denumerable set of Copeland transformations. As $F(t)$ is a monotonic function, it has at most a denumerable set of discontinuities. Let the set G consist of all those intervals I whose end points are either rational points or points of discontinuity of $F(t)$. Now let the set D consist of all possible Copeland transformations based only on intervals belonging to G. D is then a denumerable set, and almost every sequence x is regular under every transformation of D. But such a sequence is regular under all Copeland transformations, since, for any interval I, it is possible to select a sequence of intervals $I_1, I_2, \cdots$, all belonging to G, such that

$$\lim_{m\to\infty}\int_{I_m} dF(t+0) = \int_I dF(t+0).$$

8. **General remarks.** Although the theorems of Wald([24]) are more general than those of this paper in that they apply to *all* selections, no measure theory is developed for appraising sets of sequences, and transformations other than selections are not considered. Feller([25]) avoids the assumption of absolute continuity and the limitation to measurable selections by constructing a modified (and somewhat arbitrary) measure function in terms of which all configuration functions are measurable. It is not known whether this principle can be extended to transformations other than selections.

([24]) Wald [1].
([25]) Feller [1].

There are certain interesting non-endometric transformations, which nevertheless possess the regularity property of Theorem 2. No necessary and sufficient condition for this property is at present known. Nor is it known whether the general (non-symmetric) configuration transformation can be replaced by a set of operations of the von Mises types.

References

E. Borel

1. *Traité du calcul des probabilités et de ses applications*, vol. 2, no. 1, chap. 1, Paris, 1926.

A. H. Copeland

1. *A matrix theory of measurement*, Math. Zeit. vol. 37 (1933) pp. 542–555.

J. L. Doob

1. *Note on probability*, Ann. of Math. vol. 37 (1936) pp. 363–367.

W. Feller

1. *Über die Existenz von sogenannten Kollektiven*, Fund. Math. vol. 32 (1939) pp. 87–96.

T. N. E. Greville

1. *Invariance of the admissibility of numbers under certain general types of transformations*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 410–425.

A. Kolmogoroff

1. *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Ergebnisse Mathematische, vol. 2, no. 3, Berlin, 1933.

Z. Lomnicki and S. Ulam

1. *Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités*. I. *Variables indépendantes*, Fund. Math. vol. 23 (1934) pp. 237–278.

R. von Mises

1. *Vorlesungen aus dem Gebiete der angewandte Mathematik*, vol. 1, *Wahrscheinlichkeit*, Leipzig, 1931.
2. *On the foundations of probability and statistics*, Ann. Math. Statist. vol. 12 (1941) pp. 191–205.
3. *On the probabilities in a set of games and the foundation of probability theory*, unpublished manuscript.

K. Popper

1. *Logik der Forschung*, Vienna, 1935.

H. Reichenbach

1. *Les fondements logiques du calcul des probabilités*, Annales de l'Institut Henri Poincaré vol. 7 (1937) pp. 267–348.

J. Ville

1. *Étude critique de la notion de collectif*, Paris, 1939.

A. Wald

1. *Die Widerspruchsfreiheit des Kollektivsbegriffes der Wahrscheinlichkeitsrechnung*, Ergebnisse eines mathematischen Kolloquiums vol. 8 (1935–1936) pp. 38–72.

Bureau of the Census,
Washington, D. C.